TUM-HEP-352/99
MPI-PhT/99-24
June 1999

The Nielsen Identities of the SM and the definition of mass

P. Gambino[a] **and P.A. Grassi**[b]

[a] *Technische Universität München,*
Physik Dept., D-85748 Garching, Germany

[b] *Max Planck Institut für Physik (Werner-Heisenberg-Institut),*
Föhringer Ring 6, D-80805 Munich, Germany

Abstract

In a generic gauge theory the gauge parameter dependence of individual Green functions is controlled by the Nielsen identities, which originate from an enlarged BRST symmetry. We give a practical introduction to the Nielsen identities of the Standard Model (SM) and to their renormalization and illustrate the power of this elegant formalism in the case of the problem of the definition of mass. We prove to all orders in perturbation theory the gauge-independence of the complex pole of the propagator for all physical fields of the SM, in the most general case with mixing and CP violation. At the amplitude level, the formalism provides an intuitive and general understanding of the gauge recombinations which makes it particularly useful at higher orders. We also include in an appendix the explicit expressions for the fermionic two-point functions in a generic R_ξ gauge.

1 Introduction

Considering the subtle cancellations between various contributions necessary to make physical observables gauge-parameter independent, it is perhaps not surprising that the variation of individual Green functions with respect to the gauge-fixing parameters are governed by symmetry relations which guarantee precisely such cancellations. Formally, these relations can be shown to follow from an enlarged BRST symmetry in which the gauge parameters also undergo a BRST transformation [1, 2]. They are non-linear identities of the same kind of the Slavnov-Taylor Identities (STI), satisfied by Green functions at arbitrary external momenta, and are usually called Nielsen identities, after the seminal paper [3] in which they were first presented.

The power of this technique lies in the possibility of factorizing the gauge parameter dependence in terms of new objects, the Green functions of BRST sources associated to the gauge parameters. This factorization allows for efficient and elegant checks of higher order calculations. Furthermore, in the case of gauge-independent quantities, the gauge cancellations emerge from the recombination between these new objects and can be verified without an explicit evaluation of multi-loop diagrams. As we will see in the following, the mechanism of gauge recombination is revealed in great simplicity in the case of physical amplitudes, where the cancellations formally occur independently of the perturbative expansion.

The Nielsen identities are well known to field theory experts and have been extensively used in the study of the effective potential [3, 4] and in high temperature field theory [5]. Recently, they have also been studied in the context of the Abelian Higgs model [6] and of Yang-Mills theories [7] with background fields. Our purpose in this paper is to introduce the Nielsen identities of the full Standard Model (SM) and to study their renormalization. Apart from their clear relevance to conceptual problems like the gauge-invariant definition of renormalized parameters and the identification of gauge-invariant objects like effective charges, we believe they also provide a useful tool for multi-loop calculations both in the electroweak SM and in QCD. Throughout the paper, we will proceed in a pedagogical way and we will mainly concentrate on a specific physical problem, i.e. the gauge-dependence of the complex pole of the propagator for the physical fields of the SM.

The idea behind the Nielsen identities is simple: the variation of the classical action with respect to a gauge parameter coincides with the BRST variation of a local polynomial in the fields. This is clearly necessary in order to guarantee the gauge-independence of physical observables. For example, the variation of an S-matrix element with respect to the gauge parameters corresponds to the insertion of the BRST variation of a local term between physical states, which is known to vanish. The Nielsen identities implement this simple idea at the quantum level. Our starting point is the Nielsen identity for the reduced generating functional $\mathbf{\Gamma}$ [2, 3],

$$\frac{\partial}{\partial \xi}\mathbf{\Gamma} = \mathcal{S}_{\Gamma}\left(\frac{\partial}{\partial \chi}\mathbf{\Gamma}\right), \tag{1}$$

where $\chi = s\,\xi$ is the BRST source associated to a generic gauge parameter ξ, s is the classical BRST generator, and $\mathcal{S}_\Gamma$ its quantum counterpart, i.e. the Slavnov-Taylor operator whose definition is recalled in Appendix A. The use of the reduced functional, also defined in App. A, in place of the standard generating functional of proper functions is merely a technical detail: in the case of linear gauges, it allows us to write STI and Nielsen identities in a more compact way without modifying the Green functions of the physical fields. The 1PI Green functions of the theory are obtained differentiating $\mathbf{\Gamma}$ with respect to some of the SM fields. Differentiation of Eq. (1) therefore gives the gauge-dependence of a Green function in terms of products of other Green functions, which also contain the source χ. If the regularization is invariant, the identities are between *unrenormalized* Green functions. This is the case for dimensional regularization, as far as the ambiguity in the γ_5 definition [8] can be circumvented like in pure QCD. If instead a non-invariant regularization (like Pauli-Villars or BPHZ) is adopted, it is necessary to perform a complete renormalization in order to restore the symmetries of the theory.

In the most general case, the renormalization procedure at order n introduces several modifications to Eq. (1). First, the renormalization at order $n-1$ of the physical parameters of the SM can induce additional gauge-dependence at order n if the renormalization conditions are not chosen accordingly (we will see a few examples in the following). Second, the renormalization of the fields and of the unphysical sector and/or the regularization scheme adopted may break the BRST symmetry on which the Nielsen identities are based. As a consequence, Eq. (1) at order n is deformed and takes the generic form

$$\frac{\partial}{\partial\xi}\mathbf{\Gamma} = \mathcal{S}_\Gamma\left(\frac{\partial}{\partial\chi}\mathbf{\Gamma}\right) + \Delta, \tag{2}$$

where the symmetry breaking term Δ is a dimension four operator with zero ghost number such that $\mathcal{S}_\Gamma\,\Delta = 0$.

The investigation of the structure of Δ in Eq. (2) can be performed according to standard cohomological techniques [9–11]. Recalling that $\mathcal{S}_\Gamma^2 = 0$ if $\mathcal{S}_\Gamma\mathbf{\Gamma} = 0$, the first step consists in writing $\Delta = X + \mathcal{S}_\Gamma Y$ with $X \neq \mathcal{S}_\Gamma\Xi$. As can be intuitively understood, the part of Δ which can be expressed as the BRST variation of something else does not contribute to physical quantities. On the other hand, X does not decouple from the calculation of observables and is usually called the cohomology of the operator $\mathcal{S}_\Gamma$. In the SM, X is composed of the dimension four gauge-invariant operators with zero ghost number, each of them representing a cohomology class[1]. The coefficients of the cohomology classes of $\mathcal{S}_\Gamma$ are the physical parameters of the theory (gauge couplings, masses, and mixing parameters). Therefore, a contribution to X can be absorbed into a renormalization of some of the physical parameters p_i and we can write $X = \sum_i \beta_i^\xi \frac{\partial}{\partial p_i}\mathbf{\Gamma}$. For what concerns Y, it admits different kinds of contributions and is extensively studied in the literature [9, 13, 14]. The most general expression for (2) turns out to be

$$\frac{\partial}{\partial\xi}\mathbf{\Gamma} = (1+\rho^\xi)\,\mathcal{S}_\Gamma\left(\frac{\partial}{\partial\chi}\mathbf{\Gamma}\right) + \sum_i \beta_i^\xi \frac{\partial}{\partial p_i}\mathbf{\Gamma} + \sum_\varphi \gamma_\varphi^\xi\,\mathcal{N}_\varphi\,\mathbf{\Gamma} + \delta_t \int d^4x \frac{\delta\mathbf{\Gamma}}{\delta H(x)}. \tag{3}$$

[1] We recall that in the SM, besides the STI, some auxiliary constraints are needed to identify the gauge invariant operators. For a detailed discussion we refer to [12, 10, 11].

In this equation p_i are the renormalized parameters of the SM, β_i^ξ describes their explicit gauge dependence (or equivalently that of their corresponding counterterms), and φ is any of the physical or unphysical fields of the SM. When Eq. (3) is differentiated to obtain identities between Green functions, the operator $\mathcal{N}_\varphi$ counts the external fields, while ρ^ξ, γ_φ^ξ and δ_t parametrize the deformation of the Nielsen identity; they can be related to the renormalization of the gauge parameters, of the external fields, and of the tadpole, respectively. As in the SM with restricted 't Hooft gauge-fixing there are four gauge-fixing parameters ξ_i $(i = Z, W, \gamma, g)$ and as many sources χ_i, ρ^ξ is in fact a matrix. In the case of mixing between fields characterized by the same quantum numbers, also γ_φ^ξ and $\mathcal{N}_\varphi$ are matrices.

In spite of its complicated structure, Eq. (3) simply states that the considerable freedom we have in the choice of the renormalization conditions and of the regularization scheme has to be matched by an adequate number of terms which parametrize the potential breaking of symmetry. In most practical cases, however, the situation is much simpler. For example, a pure $\overline{\mathrm{MS}}$ subtraction in dimensional regularization implies not only $\beta_i^\xi = 0$, because the renormalized parameters are guaranteed to be gauge-independent [15], but also $\Delta = 0$, as long as the definition of γ_5 in d dimensions can be avoided. Similarly, in most non-minimal renormalization schemes the renormalized parameters are defined in terms of physical quantities (on-shell masses, the fine structure constant etc.), so that $\beta_i^\xi = 0$ again. In the following, we will not consider problems arising from a non-invariant regularization scheme, unless explicitly stated. In this case one can always choose the renormalization of the unphysical sector so that $\Delta = 0$. For example, one important simplification leading to $\delta_t = 0$ comes from a careful treatment of the tadpoles, which we discuss in Sec. 2. Similarly, we will see in specific examples in the following that a judicious choice of renormalization of the Green functions involving the source χ maintains the form of the Nielsen identities and implies $\rho^\xi = \gamma_\varphi^\xi = 0$.

The decomposition of Δ in Eq. (2) into X and $\mathcal{S}_\Gamma Y$ becomes important in the calculation of physical observables. As we have already noted, any operator that can be expressed as the BRST variation of something else decouples from physical quantities, hence $\mathcal{S}_\Gamma Y$ is completely irrelevant to their calculation. In Sec. 7 we will consider, in particular, the gauge cancellations leading to gauge-independent physical amplitudes. Eq. (3) tells us that neither the regularization, nor the renormalization of the fields and of the unphysical parameters, can spoil the gauge independence of the amplitudes. Only $X = \sum_i \beta_i^\xi \frac{\partial}{\partial p_i}\mathbf{\Gamma}$ can make them gauge dependent [16]. In other words, only the renormalization of the physical parameters of the theory affects the gauge-dependence of the physical observables. Incidentally, we also notice that even if the renormalization scheme leads to gauge-dependent parameters with $\beta_i^\xi \neq 0$, it is possible to avoid the appearance of the β^ξ in Eq. (3) by a redefinition of the kind $p_i \to p_i - \int_{\xi_0}^{\xi} \beta_i^\xi(\rho) d\rho$ [13, 17].

As a demonstrative ground for the technique of the Nielsen identities we have chosen the problem of the definition of mass in the SM. This is an important and non-trivial issue which recently has received renewed attention [18–21], prompted in part by the high precision measurements of the Z^0 mass at LEP and SLC. In particular, what makes the perturbative definition of the parameters associated to unstable fields a delicate and intriguing problem is the interplay between the phenomenon of resonance (which goes beyond perturbation theory as it implies the Dyson summation of an infinite number of diagrams) and the perturbative implementation of gauge symmetry. Although it has been shown long ago [22] that unstable particles are compatible with unitarity and causality, it is still worth investigating how these concepts reconcile with the underlying symmetries in the case of a full-fledged gauge-theory like the SM. We prove to all orders in perturbation theory and for all physical fields of the SM that the position of the complex pole of the propagator is gauge independent. The proof does not depend on the way the fields are renormalized and on the gauge-fixing procedure. As a consequence of the preceding discussion, it is also independent of the renormalization conditions that fix the physical parameters, provided that they do not introduce extra gauge-parameter dependence, i.e. that $\beta_i^\xi = 0$.

We have organized the paper in the following way. In the next section we introduce our notation and study the Nielsen identities for the one-point Green functions, discussing their renormalization. In Sec. 3 we consider the case of the W boson and prove the gauge-parameter independence of the pole of its propagator. Several comments and examples here should help clarify the Nielsen identities and their renormalization. As a digression, we also consider the infrared finiteness of the W pole mass. The analysis is then extended to the case of mixing. In Sec.4 we consider the γ, Z^0 sector and derive an interesting relation for the photon correlator at $q^2 = 0$ in the SM. We then study in Sec. 5 the scalar sector and in Sec. 6 the fermionic sector. The following section is devoted to a discussion of the mechanism of gauge-cancellations in the case of four-fermion processes. Sec. 8 concludes the main text summarizing the most important points. We have collected some useful material in two appendices: in the first one we discuss some aspects of the derivation of the Nielsen identities and present the sector of the Lagrangian containing the BRST sources. In App. B, instead, we provide the full one-loop fermionic self-energies in an arbitrary R_ξ gauge. This completes the work of Ref. [23], where the one-loop gauge dependence of the basic electroweak corrections has been considered.

2 Tadpoles

As a preliminary step in our analysis, we consider in this section the gauge-parameter dependence of the tadpoles. This is a very simple case and provides a first introduction to the use of the Nielsen identities; it also allows us to set the notation we will be using in the rest of the paper. Because of the close connection between the mass and the tadpole renormalizations, the results of this section will be necessary in all subsequent applications.

We denote by $\mathbf{\Gamma}^{(n)}_{\varphi_1\varphi_2,...}(p_1, p_2, ...)$ the 1PI Green function of $\varphi_1, \varphi_2, ...$ at the n-loop level. φ_i can be any physical or unphysical field of the SM in a general covariant R_ξ gauge, as well as any of the sources γ_{φ_i}, χ_j associated to the BRST variation of φ_i and of the gauge parameter ξ_j. $\mathbf{\Gamma}_{\varphi_1\varphi_2...}$ can be expressed as functional derivatives of the generating

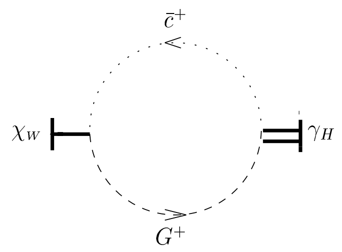

Figure 1: One-loop diagram contributing to $\mathbf{\Gamma}_{\chi\gamma_H}$.

functional, the effective action $\mathbf{\Gamma}$, with respect to the fields and sources $\varphi_1, ..., \varphi_m$,

$$\mathbf{\Gamma}_{\varphi_1 \ldots \varphi_m}(p_1, ..., p_m) = \left. \frac{\delta^m \mathbf{\Gamma}}{\delta\varphi_1(p_1) \ldots \delta\varphi_m(p_m)} \right|_{\varphi_i = 0}. \tag{4}$$

Notice that the exchange of two fermionic indices leads to a change in sign. We also adopt the short-hand notation ∂_ξ for the partial derivative with respect to a generic gauge parameter ξ, whose associated source is generically called χ. Some details concerning the action of the Slavnov-Taylor operator $\mathcal{S}_\Gamma$ on $\mathbf{\Gamma}$ and the complete source Lagrangian are given in App. A.

We are interested in the gauge parameter variation of the one-point function of the physical Higgs field, H. Differentiating both sides of Eq. (1) with respect to H and taking into account Eq. (A3), we obtain for the unrenormalized tadpole amplitude

$$-\partial_\xi \mathbf{\Gamma}_H(0) = \mathbf{\Gamma}_{\chi\gamma_H H}(0)\, \mathbf{\Gamma}_H(0) + \mathbf{\Gamma}_{\chi\gamma_H}(0)\, \mathbf{\Gamma}_{HH}(0). \tag{5}$$

All the external momenta are zero and we will drop them in the following of this section. As χ is the source associated to a gauge parameter, it is a Grassman variable which does not depend on the space-time and does not carry any momentum. In deriving Eq. (5), we have used the fact that the χ's and the γ's have ghost number equal to $+1$ and -1, respectively, and that non-vanishing Green functions must conserve the ghost charge. We have also used CP conservation to avoid, for instance, the appearance of H-G^0 mixing in higher orders. This assumption will be relaxed later. We now observe that the tree level Green functions $\mathbf{\Gamma}^{(0)}_{\chi\gamma_H H}$ and $\mathbf{\Gamma}^{(0)}_{\chi\gamma_H}$ vanish, as a consequence of the Feynman rules given in App. A. We also have $\mathbf{\Gamma}^{(0)}_H = 0$ by construction, while $\mathbf{\Gamma}^{(0)}_{HH}(0) = -M_H^2$. Expanding Eq. (5) at the one-loop level, one therefore finds

$$\partial_\xi \mathbf{\Gamma}^{(1)}_H = M_H^2\, \mathbf{\Gamma}^{(1)}_{\chi\gamma_H}, \tag{6}$$

where the last term is logarithmically divergent. It is straightforward to compute $\mathbf{\Gamma}^{(1)}_{\chi\gamma_H}$ using the Lagrangian given in App. A. Only diagrams of the kind displayed in Fig.1 contribute and we recover the gauge dependence of $\mathbf{\Gamma}^{(1)}_H$ given in Eqs.(11,12) of [23]. Beyond one-loop, however, both terms in the r.h.s of Eq. (5) contribute.

We now consider how the renormalization procedure affects this result. Our starting point is Eq. (3). First, we will assume that all β_i^ξ vanish. The γ_φ^ξ term in Eq. (3) could give a contribution proportional to the tadpole itself, which would modify the one-loop result in Eq. (6), unless the tadpole is set to zero. Indeed, the standard renormalization of the tadpole consists in setting it to zero at each perturbative order. This corresponds to minimizing the effective potential at each order [24]. Clearly, this is not the only possible choice, but its physical meaning is transparent and it is generally adopted because it simplifies the calculations. Incidentally, it is interesting to see that the tadpole counterterm is generated by the BRST variation of a local counterterm:

$$\mathbf{\Gamma}^{CT} = \delta T\, \mathcal{S}_{\Gamma_0} \left(\int d^4x\, \gamma_H \right) = \delta T \int d^4x\, \frac{\delta \mathbf{\Gamma}_0}{\delta H(x)}, \tag{7}$$

where $\mathbf{\Gamma}_0$ is the tree level action and δT the coefficient of this counterterm. It follows from Eq. (7) that a renormalization of the tadpole amplitude induces a shift proportional to δT in the mass parameters of all the SM fields. The previous equation uncovers also the unphysical nature of the renormalization of the tadpole, as the BRST variation $\mathcal{S}_{\Gamma_0}$ of a local object do not contribute to physical amplitudes. This renormalization is therefore purely conventional, but it will become clear soon that not all choices are equally convenient when one considers higher order contributions.

Eq. (7) is proportional to the last term of Eq. (3), which clearly accounts for a deformation of the Nielsen identity due to the renormalization of $\mathbf{\Gamma}_{\chi\gamma_H}$. We can see that explicitly by differentiating Eq. (3) and then setting the tadpole to zero:

$$(1+\rho^\xi)\, \mathbf{\Gamma}_{\chi\gamma_H} + \delta_t = 0. \tag{8}$$

It then follows that the *requirement* $\delta_t = 0$, i.e. the requirement that the Nielsen identity be not deformed, implies

$$\mathbf{\Gamma}^{(n)}_{\chi\gamma_H} = 0 \tag{9}$$

at any order n of perturbation theory.

In the presence of CP violation, another tadpole amplitude emerges in the SM, connected to the vacuum expectation value of the CP-odd neutral would-be Goldstone boson, G_0. As the CP violation in the SM is confined to the fermionic sector, this will happen only at higher orders. In extended models, any neutral scalar field with zero ghost charge could develop a vacuum expectation value through radiative corrections.

Without gauge interactions, the Goldstone bosons are massless because of the spontaneous symmetry breaking mechanism; at higher orders, this is enforced by the Ward-Takahashi Identities (WTI). In a non-abelian gauge theory the WTI are replaced by the STI of Eq. (A3). Upon differentiation with respect to the neutral ghost field c^Z, Eq. (A3) yields

$$\left.\frac{\delta\, \mathcal{S}_\Gamma(\mathbf{\Gamma})}{\delta c^Z(0)}\right|_{\varphi=0} = \mathbf{\Gamma}_{c^Z\gamma_0}\, \mathbf{\Gamma}_{G_0} + \mathbf{\Gamma}_{c^Z\,\gamma_H} \mathbf{\Gamma}_H = 0. \tag{10}$$

To derive the previous equation, we have used Eqs. (A3) and (A6) and the fact that one-point functions are not vanishing only for neutral scalars with zero ghost number. As can be seen from Eq. (A6), $\mathbf{\Gamma}^{(0)}_{c^Z\gamma_0}$ differs from zero already at the tree level, in which case it is proportional to v, the Higgs v.e.v.. From Eq. (10) it then follows that the vanishing of the CP-even tadpole $\mathbf{\Gamma}_H^{(n)}$ implies the vanishing of the CP-odd tadpole $\mathbf{\Gamma}_{G_0}^{(n)}$ at any order. Moreover, in the presence of CP violation a term $\delta_t^{CP} \int d^4x \, \delta\mathbf{\Gamma}/\delta G_0(x)$ should be added to Eq. (3). Using the STI for the two-point functions and the analogous of Eq. (5), and requiring $\delta_t^{CP} = 0$ one then finds that Eq. (9) is also valid, together with $\mathbf{\Gamma}_{\chi\gamma_0}^{(n)} = 0$.

In the case of a model with two Higgs-doublets [25], Eq. (10) takes the form

$$\mathbf{\Gamma}_{c^Z\gamma_0}\,\mathbf{\Gamma}_{G_0} + \mathbf{\Gamma}_{c^Z\,\gamma_H}\mathbf{\Gamma}_H + \mathbf{\Gamma}_{c^Z\,\gamma_h}\mathbf{\Gamma}_h + \mathbf{\Gamma}_{c^Z\,\gamma_A}\mathbf{\Gamma}_A = 0. \tag{11}$$

where H, h and A are the physical neutral Higgs fields. In order to minimize the effective potential, one needs to require the tadpoles of the physical fields H, h, A to vanish. It then follows that the tadpole of the unphysical Goldstone boson is zero (identifying a *flat direction* in the Higgs potential [26]) at any order in perturbation theory.

3 W boson

As a first application of the technique to the case of the definition of mass, we consider the case of the charged W boson, which is particularly simple because it does not involve any mixing between different fields. We split the unrenormalized inverse W propagator into its transverse and longitudinal parts

$$\mathbf{\Gamma}_{WW}^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\mathbf{\Gamma}_{WW}^{T}(q^2) + \frac{q^\mu q^\nu}{q^2}\mathbf{\Gamma}_{WW}^{L}(q^2). \tag{12}$$

Our first aim is to obtain a Nielsen identity for the transverse part of the propagator in the *unrenormalized* case. The longitudinal part will be considered in Sec. 5. Differentiating both sides of Eq. (1) with respect to W_μ^+ and W_ν^-, taking into account Eq. (A3), and setting to zero the Green functions which do not conserve the ghost charge, we obtain

$$\begin{aligned}\partial_\xi \mathbf{\Gamma}_{WW}^{T}(q) \;=\; & -\sum_\varphi \Big[\mathbf{\Gamma}_{\chi\gamma_\varphi WW}^{T}(q)\,\mathbf{\Gamma}_\varphi + \mathbf{\Gamma}_{\chi\gamma_\varphi}\,\mathbf{\Gamma}_{\varphi WW}^{T}(q) \\ & + t^{\mu\nu}\left(\mathbf{\Gamma}_{\chi\gamma_\varphi W_\mu}(q)\,\mathbf{\Gamma}_{\varphi W_\nu}(q) + \mathbf{\Gamma}_{\chi W_\nu\gamma_\varphi}(q)\,\mathbf{\Gamma}_{W_\mu\varphi}(q)\right)\Big]\end{aligned} \tag{13}$$

where $t^{\mu\nu} = g^{\mu\nu} - q^\mu q^\nu/q^2$ is the transverse projector and the superscript T indicates the transverse part of a Green function. The only non-vanishing one-point functions are the tadpoles, while $\mathbf{\Gamma}_{\chi\gamma_\varphi}$ for $\varphi = H, G_0$ describe the gauge-dependence of the tadpoles. For any other field φ, $\mathbf{\Gamma}_{\chi\gamma_\varphi} = 0$ because of charge and ghost number conservation or Lorentz invariance. The last term, on the other hand, is not zero only for $\varphi = W_\lambda^\pm$, so that we obtain, at any order in perturbation theory ($s = q^2$),

$$\partial_\xi \mathbf{\Gamma}_{WW}^{T}(s) \;=\; -\sum_{\varphi=H,G_0}\Big[\mathbf{\Gamma}_{\chi\gamma_\varphi WW}^{T}(s)\,\mathbf{\Gamma}_\varphi + \mathbf{\Gamma}_{\chi\gamma_\varphi}\,\mathbf{\Gamma}_{\varphi WW}^{T}(s)\Big] - 2\,\mathbf{\Gamma}_{\chi\gamma_W W}^{T}(s)\,\mathbf{\Gamma}_{WW}^{T}(s). \tag{14}$$

Here the terms in parenthesis represent the gauge variation of the $W \to W$ connected diagrams containing one or more tadpole insertions. Therefore, despite its appearance, Eq. (14) has a very simple meaning: the pole of the W propagator, i.e. the zero of $\mathbf{\Gamma}_{WW}^{T}(s)$,

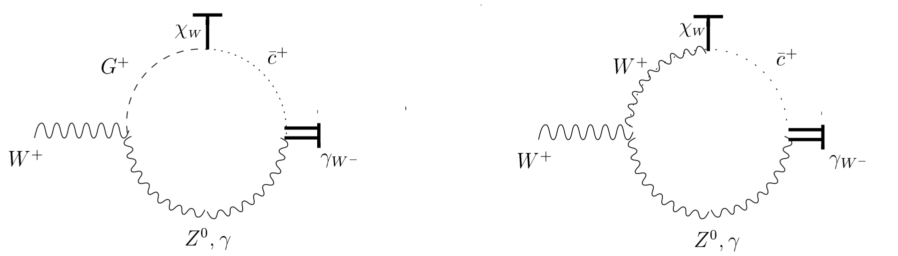

Figure 2: One-loop diagrams contributing to $\mathbf{\Gamma}_{\chi_W \gamma_W W}$.

is a gauge-independent quantity. At the one loop level, this can be easily seen: using Eq. (6) and noting that the Green functions involving χ vanish at the tree level, the previous equation reduces to

$$\partial_\xi \left[\mathbf{\Gamma}^{T,(1)}_{WW}(s) + T^{(1)}_W \right] = 2\, \mathbf{\Gamma}^{T,(1)}_{\chi\gamma_W W}(s)\,(s - M_W^2), \tag{15}$$

where $T^{(1)}_W$ is the contribution of the one-loop tadpole. The zero of the W inverse propagator is gauge-independent at $s = M_W^2$. Notice that $\mathbf{\Gamma}^{T,(1)}_{\chi\gamma_W W}(s)$ describes the gauge-dependence of the residue of the physical pole, i.e. of the on-shell wave function renormalization factor. An explicit calculation of the diagrams in Fig.2 which contribute to $\mathbf{\Gamma}^{T,(1)}_{\chi_W \gamma_W W}$ leads to the same ξ_W-dependence of $A^{(1)}_{WW}$ reported in [23]; the same happens for the $\xi_{Z,\gamma}$-dependence.

The physics of Eq. (14) is more transparent if we use its *renormalized* form. Assuming again that all $\beta^\xi_i = 0$, we set the tadpole to zero and follow the discussion of Sec. 2, imposing $\delta_t = \delta^{CP}_t = 0$: this choice implies the vanishing of $\mathbf{\Gamma}_{\chi\gamma_\varphi}$ for $\varphi = H, G_0$. We can therefore drop the terms in the parenthesis of Eq. (13), which becomes

$$\partial_\xi \mathbf{\Gamma}^T_{WW}(s) = 2 \left[-\left(1 + \rho^\xi\right) \mathbf{\Gamma}^T_{\chi\gamma_W W}(s) + \gamma^\xi_W \right] \mathbf{\Gamma}^T_{WW}(s), \tag{16}$$

with $\mathbf{\Gamma}^T_{\chi\gamma_{W^+} W^-} = \mathbf{\Gamma}^T_{\chi\gamma_{W^-} W^+}$. The significance of Eq. (16) is that a gauge invariant and self-consistent normalization condition on $\mathbf{\Gamma}^T_{WW}(s)$ can *only* be given at the location of the pole of the propagator. Defining the latter by

$$\mathbf{\Gamma}^T_{WW}(\bar{s}_W) = 0, \tag{17}$$

we see that Eq. (16) leads to $\partial_\xi \mathbf{\Gamma}^T_{WW}(s)|_{s=\bar{s}_W} = \partial_\xi(\mathbf{\Gamma}^T_{WW}(\bar{s}_W))$, which in turn implies that *the location $\bar{s}_W$ of the complex pole of the propagator is gauge-independent at any order in perturbation theory.* This is a remarkably non-trivial result of perturbation theory, as it concerns the parameters that describe the non-perturbative phenomenon of resonance. The mass parameter m_W and the width parameter Γ_W defined by $\bar{s}_W = m_W^2 - i\, m_W \Gamma_W$ are gauge independent quantities and m_W can be adopted as renormalized W mass.

Beyond one-loop the renormalization condition Eq. (17) becomes crucial in order to ensure the gauge-independence of the renormalized mass [18, 20]. Consider for instance the case in which the mass of the W boson is defined through

$$\mathrm{Re}\, \mathbf{\Gamma}^T_{WW}(M_W^2) = 0; \tag{18}$$

the W mass counterterm is then $\mathrm{Re}\,\mathbf{\Gamma}_{WW}(M_W^2)$. This is the conventional approach to one-loop mass renormalization [27–29]. Taking the real part of Eq. (16) at $s = M_W^2$, expanding it at two-loop, and setting $\rho^\xi = \gamma_\varphi^\xi = 0$ for simplicity, we obtain

$$\partial_\xi \mathrm{Re}\,\mathbf{\Gamma}^{T(2)}_{WW} \quad = \quad -2\,\mathrm{Re}\,\mathbf{\Gamma}^{T(1)}_{\chi\gamma_W W}\mathrm{Re}\,\mathbf{\Gamma}^{T(1)}_{WW} + 2\,\mathrm{Im}\,\mathbf{\Gamma}^{T(1)}_{WW}\ \mathrm{Im}\,\mathbf{\Gamma}^{T(1)}_{\chi\gamma_W W}$$

where all terms are evaluated at $q^2 = M_W^2$. Using the normalization condition Eq. (18), we see that the last term is left over, so that Eq. (16) is not satisfied. A consequence of this is that the mass definition implied by Eq. (18) is gauge-parameter dependent beyond one-loop [20]. As the imaginary part in the last term of the previous equation originates from gauge-dependent thresholds, there exists a class of gauges where $\mathrm{Im}\,\mathbf{\Gamma}^{T(1)}_{\chi\gamma_W W}(M_W^2)$ vanishes (Cf. Fig. 2) and for which the gauge parameter dependence of M_W is only apparent at the three loop level [20]. The actual difference between the two mass definitions, $\Delta M^2 = \mathrm{Re}[\mathbf{\Gamma}_{WW}(\bar{s}_W) - \mathbf{\Gamma}_{WW}(M_W^2)]$, can be evaluated expanding $\mathbf{\Gamma}_{WW}$ in powers of $|\bar{s}_W - M_W^2| \approx \Gamma_W M_W = O(g^2)$ up to $O(g^4)$. The result is $\Delta M^2 \approx M_W \Gamma_W \mathrm{Im}\,\mathbf{\Gamma}^{(1)'}_{WW}(M_W^2)$, which can be explicitly calculated by differentiating Eq. (15) and is clearly gauge parameter dependent.

A comment on the factor γ_W^ξ is now in order. As explained in the introduction, this term originates from the potential deformation of the Nielsen identity by the regularization and/or the renormalization procedure. Let us consider, for instance, dimensional regularization. As the regularization is invariant (with the proviso mentioned in the Introduction) there is no contribution to γ_W^ξ of this origin. However, there is considerable freedom in the choice of both the wave function renormalization of the W field and the renormalization of $\mathbf{\Gamma}^T_{\chi\gamma_W W}(s)$. In case they do not respect the Nielsen identity, γ_W^ξ compensates for its breaking. Let us consider, for ex., the following two procedures. A first possibility is to adopt a minimal subtraction ($\overline{\mathrm{MS}}$ scheme) for both the wave function renormalization of the W and $\mathbf{\Gamma}^T_{\chi\gamma_W W}(s)$. It should be clear that in this case $\gamma_W^\xi = 0$. A second possibility consists in using the on-shell scheme for the W field rescaling. If we now insist in using a minimal subtraction for $\mathbf{\Gamma}^T_{\chi\gamma_W W}(s)$, Eq. (14) is not satisfied by the finite parts of the counterterms, leading to a factor $\gamma_W^\xi = \mathbf{\Gamma}^T_{\chi\gamma_W W}(M_W^2)|_{\overline{\mathrm{MS}}} = \frac{1}{2}\mathbf{\Gamma}^{T'}_{WW}(M_W^2)|_{\overline{\mathrm{MS}}}$, where the subscript $\overline{\mathrm{MS}}$ means that only the finite part of this Green function is considered. Similar considerations apply to ρ^ξ, which appears first at the two-loop level and is related to the renormalization of the gauge-fixing parameters.

The renormalization condition (18) is a non-trivial example of definition of a physical parameter in a gauge-dependent way: beyond one-loop it induces $\beta^\xi_{M_W} \neq 0$. A possible source of confusion, however, is the interplay of mass and tadpole renormalization. To make this point clear, it is sufficient to keep the discussion at the one-loop level. From Eq. (15) we know that the W mass counterterm $\delta M_W^2 = \mathbf{\Gamma}^{T(1)}_{WW}(M_W^2) + T_W^{(1)}$ is gauge-independent. A tadpole renormalization according to Sec. 2, however, eliminates $T_W^{(1)}$ from the previous expression and makes δM_W^2 gauge-dependent. Nevertheless, we still have $\beta^\xi_{M_W} = 0$. This is a consequence of the unphysical character of the tadpole renormalization. What is essential here is that the renormalization condition which fixes the physical parameter M_W be gauge-independent, as is the case for Eq. (17) and not for Eq. (18). Whatever the choice of renormalization for the tadpole, this guarantees $\beta^\xi_{M_W} = 0$.

Two simple practical applications follow from Eq. (16), and we report them as illustrations of the technique. First, we can consider the dependence of the W self-energy on the QCD gauge-parameter ξ_g. Setting β_i^ξ, ρ^ξ, and γ_W^ξ to zero, it is controlled by $\mathbf{\Gamma}^T_{\chi_g\gamma_W W}$ only. However, the ghost charge associated to the QCD gauge group and the one associated to the SU(2) group must be conserved independently of each other. Therefore, $\mathbf{\Gamma}^T_{\chi_g\gamma_W W} = 0$ at any order, which implies that the W two-point function does not depend on the gluon gauge parameter, as verified in actual calculations at two and three loops [30]. The second application concerns the contributions to the W self-energy which are leading in an expansion in the heavy top quark mass. At the one-loop level, they are trivially gauge-independent, like all the fermionic contributions. At higher orders, one can use the fact that $\mathbf{\Gamma}^T_{\chi\gamma_W W}(s)$ is only logarithmically divergent to show that the gauge dependence of the heavy top expansion of $\mathbf{\Gamma}_{WW}$ starts at the next-to-leading order. Again, this is not surprising, because the leading contributions in M_t can be obtained in the framework of a Yukawa Lagrangian where the heavy fermions only couple to the Higgs boson and to the longitudinal components of the gauge bosons. This Lagrangian, which corresponds to the *gaugeless limit* of the SM [31], does not require gauge-fixing.

Infrared finiteness of the W mass

The complex pole definition of mass based on Eq. (17) avoids also infrared (IR) problems at higher orders in perturbation theory. It has been shown in Ref. [20] that the use of the normalization condition of Eq. (18) leads to severe IR divergences in a class of higher order graphs containing the photon when the external momentum approaches the mass-shell of the W. As a consequence, in the resonance region, $|s - M_W^2| \lesssim M_W \Gamma_W$, the perturbative series fails to converge, while it was found that the pole mass definition avoids all these pathologies. The origin of the problem is similar to the one of the gauge-dependence of the mass parameter defined by Eq. (18) and is related to the need to take into account the imaginary part of $\mathbf{\Gamma}^T_{WW}$ in the renormalization procedure.

More generally, the problem is common to all particles coupled to massless quanta, independently of whether they are stable or not, and concerns the perturbative description of the resonance region. For instance, in pure QCD it is well-known [32] that at two-loop order the two-point function of a massive quark is IR divergent at $q^2 = m_q^2$ unless the quark mass is renormalized on the pole. In Ref. [21] it was shown that this property persists at all orders in QCD, namely that the perturbative pole mass of the quark in QCD is infrared safe (or finite). In the following we would like to approach the case of the W boson from a slightly different point of view, along the lines of [21], generalizing some of the results of Ref. [20]. We will show that the complex pole mass of the W is IR safe at any order in perturbation theory, namely that the renormalization condition of Eq. (17) does not lead to IR divergences in the resonance region of the W boson, nor to pathologies in the perturbative expansion. In that respect, the presence of the width does not alter the discussion in a relevant way.

A convenient tool to analyze the IR properties of the W self-energy from a perturbative point of view are the renormalized Schwinger-Dyson equations (see e.g. [33]). These

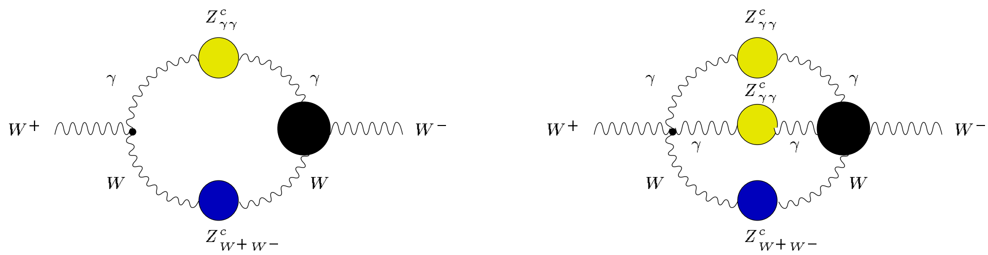

Figure 3: Schwinger-Dyson equation for the W two-point function. The blobs on the internal lines represent connected propagators (chains of bubbles), while the blob on the vertex represents a one-particle irreducible Green function.

equations provide a simple iterative way to define the higher order graphs in terms of sub-diagrams. In the case of the W boson there are only two topologies containing the photon which should be considered, as they contain thresholds at $s = M_W^2$ and can lead at higher orders to IR problems. Their Schwinger-Dyson equations are graphically depicted in Fig. 3. Diagrams with gauge-dependent threshold (like those with a charged Goldstone boson in place of the W) and with thresholds far away from the resonance region (like those with a Z^0 boson instead of the photon) can be discarded because their expansion around $s = M_W^2$ does not contain non-analytic terms.

We will treat explicitly only the case of the topology on the left side of Fig. 3, as the other diagram can be discussed along the same lines. In this case the Schwinger-Dyson equation has the form

$$\mathbf{\Gamma}^{(\gamma)}_{W^+_\mu W^-_\nu}(p) \sim \int d^n k \, \mathbf{\Gamma}^{(0)}_{W^+_\mu A_\rho W^-_\sigma}(k, p+k) \, Z^c_{A_\sigma A_\beta}(k) \, Z^c_{W^+_\sigma W^-_\alpha}(k+p) \, \mathbf{\Gamma}_{W^+_\alpha A_\beta W^-_\nu}(k, p), \tag{19}$$

where $\mathbf{\Gamma}^{(\gamma)}_{W^+_\mu W^-_\nu}(p)$ is the contribution to the self-energy due to the exchange of a single photon, $\mathbf{\Gamma}_{W^+_\mu A_\rho W^-_\sigma}(k, p+k)$ is the 1PI vertex, the superscript (0) indicates that the vertex is considered at the tree level, and finally $Z^c_{A_\sigma A_\beta}(k)$ and $Z^c_{W^+_\sigma W^-_\alpha}(k+p)$ are the connected propagator for the photon and for the W boson, respectively. To study the IR behavior of Eq. (19) near the mass-shell, we now consider the transverse part of the self-energy $\mathbf{\Gamma}^{(\gamma)}_{W^+_\mu W^-_\nu}(p)$ and approach the limit $p^2 \to \bar{s}_W$. We expand the propagator into the Dyson series of self-energies and tree propagators. Concerning the photon line, we recall that a convenient choice of the normalization conditions for the neutral gauge boson sector, i.e. $\mathbf{\Gamma}^T_{ZA}(0) = 0$, makes $\mathbf{\Gamma}^T_{AA}(0)$ vanish at all orders (Cf. next section). Therefore, the photon propagator $Z^c_{A_\sigma A_\tau}(k)$ is always proportional to $1/k^2$ in the limit $k \to 0$ and has IR dimension -2.

For what concerns the W propagator, the IR divergent contributions are related only to the transverse component of $Z^c_{W^+W^-}(k+p)$ because the propagator of the longitudinal components of the W boson has a gauge dependent pole. In the on-shell limit for the momentum p and for $k \to 0$, the tree level W propagators present in the Dyson series for $Z^{c,T}_{W^+W^-}(k+p)$ are linearly divergent. Therefore, expanding $Z^{c,T}_{W^+W^-}(k+p)$ around $k = 0, p^2 = \bar{s}_W$ we have

$$Z^{c,T}_{W^+W^-}(k+p)\Big|_{p^2=\bar{s}_W} \overset{k\to 0}{\simeq} \sum_n \left(\frac{1}{2p\cdot k}\right)^{n+1} \left[\mathbf{\Gamma}^T_{WW}(\bar{s}_W)\right]^n . \tag{20}$$

Here we consider only the most dangerous terms, which vanish if and only if $\mathbf{\Gamma}^{T}_{WW}(\bar{s}_W) = 0$. Under this condition, $Z^{c,T}_{W^+W^-}(k+p)\Big|_{p^2=\bar{s}_W}$ is at most linearly divergent in the IR limit. If, on the other hand, Eq. (17) is not satisfied, severe IR divergences appear in each order. The situation is not much improved if we move off the pole position in the resonance region. Indeed, in this case the W width acts as an IR regulator in the denominator of Eq. (20), but leads to a series where the denominator $1/(s-\bar{s}_W) \approx O(1/g^2)$ spoils the convergence of the perturbative expansion in the resonance region [20].

The last information we need concerns the behavior of the vertex $\mathbf{\Gamma}^{T}_{W^+A_\beta W^-}(k,p)$ (T refers to the transverse components of the W bosons) around $k=0, p^2 = M_W^2$. By analiticity and dimensional analysis, the vertex functions can be at most logarithmically divergent in the limit $k \to 0$ (this can also be verified exploiting the STI together with a proper use of the renormalization conditions). Having IR dimension -3, it follows by power counting that Eq. (19) does not lead to IR divergences when the integral in the internal momentum k is performed around $k=0$.

In summary, we have seen that the pole mass of the W boson, defined by Eq. (17), is IR safe to all orders in perturbation theory and that only if this definition is adopted a perturbative description of the resonance region is possible.

4 The $Z - \gamma$ system

The main difference between the case of the W boson and the one of the neutral vector bosons is the presence of mixing. We now directly use Eq. (3), assuming all $\beta_i^\xi = 0$ and setting $\rho^\xi = 0$ for ease of notation (doing otherwise would not modify our results). Following the same steps as in the derivation of Eq. (16), we find for $i,j = A,Z$

$$\partial_\xi \mathbf{\Gamma}^{T}_{ij}(s) = -\sum_{k=A,Z}\left[\left(\mathbf{\Gamma}^{T}_{\chi\gamma_k i}(s) - \gamma^\xi_{ik}\right)\mathbf{\Gamma}^{T}_{kj}(s) + \mathbf{\Gamma}^{T}_{ik}(s)\left(\mathbf{\Gamma}^{T}_{\chi\gamma_k j}(s) - \gamma^\xi_{kj}\right)\right], \tag{21}$$

where γ^ξ_{ij} is the deformation induced by the possible mismatch between the wave function renormalization matrix Z_{ij} and the renormalization of $\mathbf{\Gamma}_{\chi\gamma_i j}$. We recall that $\mathbf{\Gamma}^{T}_{ik}(s)$ is a symmetric matrix. We now consider the quantity

$$\mathcal{D}^{T}_{AZ}(s) = \det\begin{pmatrix} \mathbf{\Gamma}^{T}_{AA}(s) & \mathbf{\Gamma}^{T}_{AZ}(s) \\ \mathbf{\Gamma}^{T}_{ZA}(s) & \mathbf{\Gamma}^{T}_{ZZ}(s)\end{pmatrix}, \tag{22}$$

which appears in the denominator of the propagators of the photon-Z^0 system (see for ex. [29]). If we are interested in the analytic structure of neutral current amplitudes in the typical configuration of a high-energy collider, where external fermion masses can be neglected, $\mathcal{D}^{T}_{AZ}(s)$ is what we need to investigate. It is straightforward to derive

$$\partial_\xi \mathcal{D}^{T}_{AZ}(s) = -2\left(\mathbf{\Gamma}^{T}_{\chi\gamma_A A}(s) - \gamma^\xi_{AA} + \mathbf{\Gamma}^{T}_{\chi\gamma_Z Z}(s) - \gamma^\xi_{ZZ}\right)\mathcal{D}^{T}_{AZ}(s). \tag{23}$$

This tells us that the zeros of $\mathcal{D}^{T}_{AZ}$ identify gauge-independent quantities. On the other hand, we know from the STI that $\mathcal{D}^{L}_{AZ}(0) = 0$ (see for ex. [28]; Ref. [11] considers also the case of CP violation) which in turn implies by analyticity $\mathcal{D}^T_{AZ}(0) = 0$. This result ensures the existence of a massless state, the photon. $\mathcal{D}^T_{AZ}$ has, however, another zero, corresponding to the Z^0 complex pole, at $q^2 = \bar{s}_Z$. As in the case of the W boson, this result implies that the position of the complex pole is a gauge independent quantity and that the only self-consistent normalization condition for the Z^0 mass is the one given in analogy to Eq. (17). With the exception of the IR problems, all the discussion on the W mass applies directly to the case of the Z^0 boson mass [18]. A Ward Identity similar to the Nielsen identity of Eq. (16) has been applied in [19] to the case of the Z^0 resonance, to the same avail.

Another interesting application of Eq. (21) concerns the photon correlator at $q^2 = 0$. As is well known [28], if the renormalization condition $\mathbf{\Gamma}^T_{AZ}(0) = 0$ is imposed, the result $\mathcal{D}^T_{AZ}(0)$ that we have used above implies $\mathbf{\Gamma}^T_{AA}(0) = 0$. In this case the only object that enters the standard electric charge renormalization is the derivative wrt q^2 of the photon two-point function calculated at $q^2 = 0$. It is straightforward to verify from Eq. (21) that this object is gauge-independent at all orders. Imposing the condition $\mathbf{\Gamma}^T_{AZ}(0) = 0$ and setting $\gamma^\xi_\varphi = 0$ in the expression of $\partial_\xi \mathbf{\Gamma}^T_{AZ}(0)$, we obtain the constraint $\mathbf{\Gamma}^T_{\chi^A \gamma_Z}(0) = 0$. But we know from Eq. (A7) that $\mathbf{\Gamma}^T_{\chi^A \gamma_A}(q)$ is proportional to $\mathbf{\Gamma}^T_{\chi^A \gamma_Z}(q)$ because the composite operators coupled to γ_A and to γ_Z are the same up to linear terms in the ghost fields which do not contribute to the above Green functions. Therefore, we have $\mathbf{\Gamma}^T_{\chi^A \gamma_A}(0) = 0$. We can now differentiate $\partial_\xi \mathbf{\Gamma}^T_{AA}$ wrt s and evaluate it at $s = 0$. Using the various constraints we have obtained, we immediately derive

$$\partial_\xi \left. \frac{\partial}{\partial s} \mathbf{\Gamma}^T_{AA}(s) \right|_{s=0} = 0. \tag{24}$$

Notice that no renormalization condition on the derivative $\frac{\partial}{\partial s} \mathbf{\Gamma}^T_{AA}|_{s=0}$ has been imposed, so one can adopt, for instance, a minimal subtraction. This interesting and non-trivial result shows that under the condition $\mathbf{\Gamma}^T_{AZ}(0) = 0$ and at $s = 0$ there exists in the full SM something analogous to what happens in QED, where the vacuum polarization of the photon is gauge-independent for any s (see for ex. Ref. [34]). An alternative derivation of Eq. (24) can be obtained starting from the physical photon-electron amplitude at $s = 0$, proceeding along the lines of the discussion of Sec. 7, and taking the gauge-independence of the on-shell amplitude for granted.

5 The scalar sector

In the previous section we have studied a first example of mixing. Indeed, mixing occurs in several other cases in the SM and in most of its extensions; all can be treated in a way very similar to the $\{\gamma, Z\}$ case discussed above. In this section, we first consider the matrix $\mathbf{\Gamma}^\phi(s)$ of the two point functions relative to the scalar fields $\phi = \{\phi_1, \phi_2, ..., \phi_n\}$ in the general case of mixing and show that the gauge dependence of its determinant follows an equation analogous to Eq. (23), if the rank of $\mathbf{\Gamma}^\phi(s)$ is equal to its dimension n. As CP violation is present in the SM, we then consider the system formed by $\{A_L, Z_L, G_0, H\}$, where the subscript L denotes the longitudinal component of the vector boson fields. This system is highly constrained by the STI and we show that in this case the complex pole of the only physical field, the Higgs boson, is gauge-invariant. In an analogous way one can consider the $\{W_L^\pm, G^\pm\}$ system, which however has no physical degree of freedom and is completely constrained by the STI.

The general form of the Nielsen identity in the case of a system ϕ of fields characterized by the same conserved quantum numbers can be obtained in analogy to Eq. (21) and reads

$$\partial_\xi \mathbf{\Gamma}^\phi(s) = \Lambda(s)\, \mathbf{\Gamma}^\phi(s) + \mathbf{\Gamma}^\phi(s)\, \Lambda'(s), \tag{25}$$

where we do not need to specify the matrices Λ and Λ' any further. Using $\ln \det \mathbf{\Gamma}^\phi = \mathrm{tr} \ln \mathbf{\Gamma}^\phi$ and exploiting the properties of the trace, one finds for $\mathcal{D}_\phi \equiv \det \mathbf{\Gamma}^\phi$

$$\partial_\xi \mathcal{D}_\phi(s) = \mathrm{tr}\,[\Lambda(s) + \Lambda'(s)]\, \mathcal{D}_\phi(s), \tag{26}$$

which generalizes Eq. (23) in the case the rank of $\mathbf{\Gamma}^\phi(s)$ at arbitrary s is equal to its dimensionality. In the case of n scalar fields this ensures the gauge-independence of n complex poles.

Neutral current processes are mediated by photons and Z^0, as well as by scalar fields, like G_0 and the physical Higgs. As it is well-known, the propagator matrix is obtained by inversion of the two-point function matrix and, in the process of inversion, the transverse and longitudinal components of the vector boson fields decouple. Having considered the transverse degrees of freedom in the preceding section, we can now limit ourselves to the system formed by the longitudinal components of the photon and of the Z^0 and by the Higgs and the neutral Goldstone bosons, which we denote by $S = \{A_L, Z_L, G_0, H\}$. The two point functions involving one vector boson and one scalar are defined extracting q^μ. In this way, $\mathbf{\Gamma}^S$ is the $4{\times}4$ matrix of the two-point functions of S.

The system S includes unphysical degrees of freedom. As we have noted in the introduction, even at the tree level the Green functions of unphysical fields are modified by the choice to use the reduced generating functional $\mathbf{\Gamma}$ in place of the complete functional $\mathbf{\Gamma}^c$ (see the App. A). For the purposes of this section, however, the reduced functional simplifies significantly the derivation without affecting the physical information we can extract from $\mathbf{\Gamma}^S$. In a way, this can be viewed as a consequence of the fact that the cancellation between the unphysical degrees of freedom occurs independently of the gauge-fixing sector [28, 12].

Each row of $\mathbf{\Gamma}^S$ is connected by a STI. For instance, differentiating Eq. (A3) with respect to A^μ and c^A, we obtain for the first row

$$\mathbf{\Gamma}^L_{AA}\mathbf{\Gamma}_{c_A\gamma_A} + \mathbf{\Gamma}^L_{AZ}\mathbf{\Gamma}_{c_A\gamma_Z} + \mathbf{\Gamma}_{AG_0}\mathbf{\Gamma}_{c_A\gamma_0} + \mathbf{\Gamma}_{AH}\mathbf{\Gamma}_{c_A\gamma_H} = 0. \tag{27}$$

Similar identities can be derived for the other rows, so that the STI for the two-point functions can be written as $\mathbf{\Gamma}^S V_{c_A} = 0$, where $V_{c_A} = (\mathbf{\Gamma}_{c_A\gamma_A}, \mathbf{\Gamma}_{c_A\gamma_Z}, \mathbf{\Gamma}_{c_A\gamma_0}, \mathbf{\Gamma}_{c_A\gamma_H})^T$. Since ϕ includes the unphysical components of the photon and Z^0 fields and since we have eliminated the gauge fixing sector of the tree level Lagrangian in using the reduced functional (see Eq. (A2)), it is perhaps not surprising that there is no propagator for A_L and Z_L and that $\det \mathbf{\Gamma}^S = 0$ or the rank of $\mathbf{\Gamma}^S$ is less than 4. In fact, $\mathbf{\Gamma}^S$ has another linearly independent eigenvector V_{c_Z} with zero eigenvalue, corresponding to the set of STI obtained by differentiation wrt c_Z. Therefore, the rank of $\mathbf{\Gamma}^S$ is at most 2 and that we cannot use Eq. (26) at this stage. Moreover, the sub-matrix of $\mathbf{\Gamma}^S$ identified by the indices G_0 and H has the same rank as $\mathbf{\Gamma}^S$. This can be seen by noting that V_{c_Z} and V_{c_A} can be orthogonalized in the subspace of the A^L and Z^L components because

$$\det \begin{pmatrix} \mathbf{\Gamma}_{c_A\gamma_A} & \mathbf{\Gamma}_{c_Z\gamma_A} \\ \mathbf{\Gamma}_{c_A\gamma_Z} & \mathbf{\Gamma}_{c_Z\gamma_Z} \end{pmatrix} = 1 + O(\hbar) \neq 0, \tag{28}$$

which follows from Eq. (A7). Having eliminated the unphysical longitudinal components of the vector bosons, we can now concentrate on the sub-matrix

$$\mathbf{\Gamma}^{\mathcal{H}} = \begin{pmatrix} \mathbf{\Gamma}_{G_0G_0} & \mathbf{\Gamma}_{G_0H} \\ \mathbf{\Gamma}_{HG_0} & \mathbf{\Gamma}_{HH} \end{pmatrix}, \tag{29}$$

whose rank is equal to the one of $\mathbf{\Gamma}^\phi$. Indeed, at arbitrary q^2, its rank is 2, so that Eq. (26) is satisfied. $\mathbf{\Gamma}^{\mathcal{H}}$ is very similar to the $\gamma - Z^0$ transverse mixing matrix. Even if the CP violation mixes up physical and unphysical scalar fields at high perturbative orders, it is not difficult to disentangle them taking advantage of the STI. At $q^2 = 0$ the two STI obtained by differentiating wrt $c_{A,Z}$ and G_0 imply that $\det \mathbf{\Gamma}^{\mathcal{H}}(0) = 0$. This zero is related to the G^0 field and is located at $q^2 = 0$ (in the standard R_ξ gauge it would be at $q^2 = \xi_Z M_Z^2$) as a consequence of the use of the reduced functional. The remaining zero, at $q^2 = \bar{s}_H$, corresponds instead to the physical pole of the Higgs boson and its location in the complex plane is therefore gauge-independent, as it follows from Eq. (26). A discussion of the relation between the pole mass and the conventionally renormalized mass of the Higgs boson in this case can be found in Ref. [35].

Turning back to the generic case of mixing between a set of scalar fields ϕ, we recall that the physical information contained in the matrix $\mathbf{\Gamma}^\phi$ is not restricted to the physical poles. The higher order definition of the mixing parameters is also affected by the off-diagonal elements of $\mathbf{\Gamma}^\phi$. As usual, the minimal $\overline{\text{MS}}$ renormalization automatically ensures the gauge-independence of the mixing parameters. More generally, however, it is not easy to form gauge-independent quantities that can be used to renormalize the mixing parameters and great care should be exercised in order to avoid the introduction of spurious gauge-dependence.

6 Fermions

The treatment of the fermionic sector is only slightly more involved than that of the scalar sector. Again, we consider the most general case of mixing and call $\mathbf{\Gamma}^f$ the matrix of the fermionic two-point functions, $\mathbf{\Gamma}_{\bar{f}f'}$. In the case of massless neutrinos, there is no mixing in the leptonic sector and $\mathbf{\Gamma}^{lept}$ is a diagonal matrix. As a first step, we need to decompose $\mathbf{\Gamma}^f$ into scalar pieces:

$$\mathbf{\Gamma}^f(p) = \Sigma_L(p^2)\, \not{p}\, P_L + \Sigma_R(p^2)\, \not{p}\, P_R + \Sigma_D(p^2) P_L + \Sigma_D^\dagger(p^2) P_R \tag{30}$$

where $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$ are the left and right-handed projectors. As can be seen by inverting $\mathbf{\Gamma}^f$, the relevant quantity for the fermion propagator matrix is the matrix [36]

$$K_f(p^2) = p^2 \, \Sigma_L - \Sigma_D^\dagger \, \Sigma_R^{-1} \, \Sigma_D, \tag{31}$$

where we have dropped the p^2 dependence of the Σ matrices. Since the determinant of this matrix appears in the denominator of the fermion propagators, we want to study its zeros, i.e. the zeros of the eigenvalues of K_f. We recall that by pseudo-hermiticity $\mathbf{\Gamma}^f = \gamma_0^\dagger \, \mathbf{\Gamma}^{f\dagger} \, \gamma_0$, so that $\Sigma_L^\dagger(p^2) = \Sigma_L(p^2)$ and $\Sigma_R^\dagger(p^2) = \Sigma_R(p^2)$ (this is actually true below thresholds, but it does not affect our conclusions). Hence, the matrix $K_f(p^2)$ is hermitian and can be diagonalized by means of a unitary transformation. Under the usual assumptions $\beta_i^\xi = \rho^\xi = \gamma_\varphi^\xi = 0$, the gauge-parameter dependence of $\mathbf{\Gamma}^f$ is described by a Nielsen identity which has exactly the same form as Eq. (25), apart from the fact that $\Lambda = -\mathbf{\Gamma}_{\chi \bar{f} \eta_{f'}}$ and $\Lambda' = -\mathbf{\Gamma}_{\chi \bar{\eta}_f f'}$ have a Dirac structure and undergo a decomposition analogous to Eq. (30). Again by pseudo-hermiticity, we find that in this case $\Lambda_{L,R} = (\Lambda'_{L,R})^\dagger$ and $\Lambda_D = \Lambda'_D$. It is then straightforward to verify that the components of $\mathbf{\Gamma}^f$ satisfy

$$\begin{aligned}
\partial_\xi \Sigma_L &= \Lambda_L \Sigma_D + \Sigma_L \Lambda_D + \Lambda_D^\dagger \Sigma_L + \Sigma_D^\dagger \Lambda_L^\dagger \\
\partial_\xi \Sigma_R &= \Lambda_D \Sigma_R + \Sigma_R \Lambda_D^\dagger + \Lambda_R \Sigma_D^\dagger + \Sigma_D \Lambda_R^\dagger \\
\partial_\xi \Sigma_D &= p^2 (\Lambda_R \Sigma_L + \Sigma_R \Lambda_L^\dagger) + \Lambda_D \Sigma_D + \Sigma_D \Lambda_D \ ,
\end{aligned} \tag{32}$$

from which it follows that

$$\partial_\xi K_f = K_f \, F + F^\dagger \, K_f \, , \tag{33}$$

with $F = \Lambda_D - \Lambda_R^\dagger \Sigma_R^{-1} \Sigma_D$. Without using pseudo-hermiticity, we would have $F' \neq F^\dagger$ in place of $F^\dagger$ in the previous equation. Eq. (33) is in the form of Eq. (25) and therefore $\mathcal{D}_f \equiv \det K_f$ satisfies Eq. (26). We have therefore algebraically reduced the problem in the fermionic case to the scalar one. In the case of mixing between n fermions, the gauge-parameter independence of n complex poles is thus warranted. We stress that this result holds for any choice of the fermion wave function renormalization and is independent of the renormalization of the mixing parameters (CKM matrix angles and CP violating phase).

The above proof is valid in the full SM. For what concerns pure QED and QCD, the result that the pole masses of the electron and of the quark are gauge-independent is not new and has been obtained both using the Nielsen identities [34] and in different ways [21, 37]. In QED (QCD) the situation simplifies considerably: writing $\mathbf{\Gamma}_{\bar{f}f} = B \not{p} - mA$, where m is the mass of the electron (quark), and decomposing Λ in an analogous way, we find

$$-\partial_\xi A = \frac{p^2}{m} \, B \, \Lambda_B + m \, A \, \Lambda_A; \qquad -\partial_\xi B = m \, (A \, \Lambda_B + B \, \Lambda_A) \, , \tag{34}$$

which could be tested up to $O(\alpha_s^2)$ against the general R_ξ gauge calculation of Ref. [38].

The proof of the IR finiteness of the fermions in the SM follows Ref. [21] and the final discussion in Sec. 3 and is already present *in nuce* in Ref. [20]. For completeness, in App. B we present the explicit gauge-parameter dependence of the one-loop fermionic self-energies in a general R_ξ gauge for the full SM. Remembering that Λ first occur at the one-loop level, it is straightforward to see that they satisfy the Nielsen identities Eq. (32). This completes the set of expressions given in Ref. [23] and is very useful in particular applications. For instance, Eqs. (B1-B3) have been used in Ref. [39] to discuss the gauge dependence of the one-loop definition of the CKM matrix. Indeed, as noted in the previous section, the renormalization of the mixing parameters is a delicate subject for what concerns the gauge-parameter dependence. An adequate framework for studying it is the Background Field Method [11]. In the case of the fermion mixing a comprehensive analysis has been presented in Ref. [39].

7 Application to physical amplitudes

In this section we apply the formalism of the Nielsen identities to four-fermion physical amplitudes and study the mechanism of gauge cancellations at any order in perturbation theory. Our purpose here is not to prove the gauge-independence of the physical amplitudes, a result which was accomplished in full generality long ago at the level of the generating functional [16]. We would rather like to study a specific example and carry out the analysis at an arbitrary order in perturbation theory. The use of the Nielsen identities allows us to uncover the regularities of the gauge recombinations between the different components (vertices, boxes and self-energies) in great generality. We stress the fact that the following derivation is independent of the perturbative expansion of the Green functions. In other words, if we work at order n in perturbation theory the Green functions have to be expanded up to this order, but the factorization is formally valid even if no perturbative expansion exists. At the one-loop level, a similar factorization is also accomplished diagrammatically by the Pinch Technique (PT) [40], whose extension at higher orders has however proved problematic. Unlike the PT, the Nielsen identities control only the gauge parameter variation and cannot be used to construct explicitly gauge-independent proper functions which satisfy basic requirements and tree-level-like Ward identities. However, they may prove useful in the search for the higher-order extension of the PT. The analysis of this section gives us also the opportunity to present explicitly the Nielsen identities for vertices and boxes involving fermions, which are interesting in their own respect, as they appear in most phenomenological applications.

We first consider the truncated Green function $Z^{trunc}_{\bar{I}J\bar{K}N}$ (see e.g. [33]) for a generic four fermion process $f_{\bar{I}} f_J \longrightarrow f_{\bar{K}} f_N$ and we decompose it in terms of irreducible diagrams and propagators. We will use capital and lowercase letters to denote fermions and bosonic fields (scalar as well as gauge vector bosons), respectively. Therefore, $Z^c_{\bar{I}J}$ and Z^c_{ij} are the propagator functions of fermions and bosons. Following the convention of the preceding sections, irreducible boxes and vertices are denoted by $\mathbf{\Gamma}_{\bar{I}J\bar{K}N}, \mathbf{\Gamma}_{\bar{I}Ji}$, and $\mathbf{\Gamma}_{j\bar{K}N}$. To keep the notation simple, we drop Lorentz indices and the dependence on the external momenta. The physical amplitude $\mathcal{M}_{\bar{I}J\bar{K}N}$ for our process is obtained from $Z^{trunc}_{\bar{I}J\bar{K}N}$ using the LSZ reduction formula [33], which in the case of fermion with mixing reads [28]

$$\mathcal{M}_{\bar{I}J\bar{K}N} = \lim_{on-shell} \widetilde{Z}^{1/2}_{\bar{I}I'}\, \widetilde{Z}^{1/2}_{\bar{J}'J}\, Z^{trunc}_{\bar{I}'J'\bar{K}'N'}\, \widetilde{Z}^{1/2}_{\bar{K}K'}\, \widetilde{Z}^{1/2}_{\bar{N}'N}, \tag{35}$$

where the on-shell limit includes the projection on the asymptotic states and $\widetilde{Z}$ controls the relation between the asymptotic states and the renormalized spinors:

$$\widetilde{Z}^{1/2}_{\bar{I}J}\, u_{as,J} = u_J. \tag{36}$$

The matrix $\widetilde{Z}$ can be computed from the conditions [28] (quantum equations of motion)

$$\begin{aligned} \mathbf{\Gamma}_{\bar{I}J}\, u_J(m_J) = 0; &\qquad \bar{u}_I(m_I)\, \mathbf{\Gamma}_{\bar{I}J} = 0 \\ \frac{1}{\not{p} - m_I}\, \mathbf{\Gamma}_{\bar{I}I}\, u_I(m_I) = u_I(m_I); &\qquad \bar{u}_I(m_I)\, \mathbf{\Gamma}_{\bar{I}I}\, \frac{1}{\not{p} - m_I} = \bar{u}_I(m_I), \end{aligned} \tag{37}$$

using the fact that $(\not{p} - m_J)\, u_{as}(m_J) = 0$ at any order by definition. Of course, $\widetilde{Z}$ should be decomposed in left and right-handed parts, $\widetilde{Z} = \widetilde{Z}^L P_L + \widetilde{Z}^R P_R$. Notice that the first line of Eqs. (37) implies $\det K_f = 0$ and consequently corresponds to the requirement that the mass parameters of the external fermions are renormalized on the poles of the propagators (see Sec. 6). Strictly speaking, the LSZ formalism applies only to stable external states, i.e. to the electron and neutrinos and, to a good approximation, to the muon. Nevertheless, we will consider here the general case of mixing. We also stress that the LSZ factors $\widetilde{Z}$ should not be confused with the wave-function renormalization factors for the external fields. Of course, the latter can be *chosen* by imposing Eqs. (37) together with $\widetilde{Z} = \mathbf{1}$ (*on-shell* scheme [28]), but there is in general no restriction on their choice (see also [39]) and it is even possible to avoid them altogether, in which case $\widetilde{Z}$ is divergent. Once the wave-function renormalization has been defined, for instance through a minimal subtraction, the factors $\widetilde{Z}$ can be computed from Eqs. (37).

As a first step, we consider the gauge variation of the truncated Green function $Z^{trunc}_{\bar{I}J\bar{K}N}$ In the most general case of mixing, $Z^{trunc}_{\bar{I}J\bar{K}N}$ is decomposed in the following blocks (we sum over repeated indices)

$$Z^{trunc}_{\bar{I}J\bar{K}N} \;=\; i\, \mathbf{\Gamma}_{\bar{I}J\bar{K}N} - \mathbf{\Gamma}_{\bar{I}Ji}\, Z^c_{ij}\, \mathbf{\Gamma}_{j\bar{K}N} - \mathbf{\Gamma}_{\bar{I}Ni}\, Z^c_{ij}\, \mathbf{\Gamma}_{j\bar{K}J}, \tag{38}$$

from which we obtain

$$\begin{aligned} \partial_\xi Z^{trunc}_{\bar{I}J\bar{K}N} \;=\;& i\, \partial_\xi \mathbf{\Gamma}_{\bar{I}J\bar{K}N} \\ -\;& (\partial_\xi \mathbf{\Gamma}_{\bar{I}Ji})\, Z^c_{ij}\, \mathbf{\Gamma}_{j\bar{K}N} - \mathbf{\Gamma}_{\bar{I}Ji} \left(\partial_\xi Z^c_{ij}\right) \mathbf{\Gamma}_{j\bar{K}N} - \mathbf{\Gamma}_{\bar{I}Ji}\, Z^c_{ij}\, \partial_\xi \mathbf{\Gamma}_{j\bar{K}N} \\ -\;& (\partial_\xi \mathbf{\Gamma}_{\bar{I}Ni})\, Z^c_{ij}\, \mathbf{\Gamma}_{j\bar{K}J} - \mathbf{\Gamma}_{\bar{I}Ni} \left(\partial_\xi Z^c_{ij}\right) \mathbf{\Gamma}_{j\bar{K}J} - \mathbf{\Gamma}_{\bar{I}Ni}\, Z^c_{ij}\, \partial_\xi \mathbf{\Gamma}_{j\bar{K}J}\,. \end{aligned} \tag{39}$$

We can compute the different contributions $\partial_\xi \mathbf{\Gamma}_{J\bar{I}N\bar{K}}, \partial_\xi \mathbf{\Gamma}_{J\bar{I}i}$, and $\partial_\xi Z^c_{ij}$ using the Nielsen identities. The identity for the propagator functions Z^c_{ij} and $Z^c_{\bar{I}J}$ is easily derived from the identity for the irreducible **two-point functions** $\mathbf{\Gamma}_{ij}$ and $\mathbf{\Gamma}_{\bar{I}J}$. As we have seen, the general form of the latter is

$$\partial_\xi \mathbf{\Gamma}_{ab} = -\mathbf{\Gamma}_{ac}\, \mathbf{\Gamma}_{\chi b \gamma_c} - \mathbf{\Gamma}_{\chi a \gamma_c}\, \mathbf{\Gamma}_{cb}, \tag{40}$$

where the indices a, b apply to both the bosonic and fermionic case. As usual, we have removed all tadpole amplitudes and assumed $\beta^\xi_i = 0$, $\forall i$. Concerning the ρ^ξ and γ^ξ_φ factors,

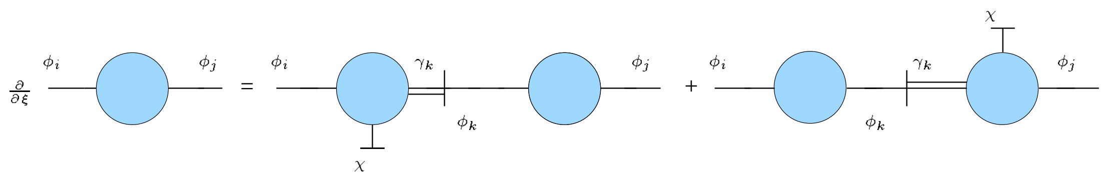

Figure 4: Nielsen identity for the two-point function $\boldsymbol{\Gamma}_{\phi_i\phi_j}$.

we avoid them here for ease of notation. However, following the discussion in Sec. 1, they are bound to drop out of the amplitude and this can be explicitly verified. In other words, although a generalization is straightforward, the following applies to a situation in which the physical parameters are renormalized in a gauge-independent way (for ex. in $\overline{\text{MS}}$) and a minimal subtraction (or no subtraction at all, as in [27]) is performed for the rest of the theory. Eq. (40) can be graphically represented in the very simple way shown in Fig. 4. Notice that the momentum flows along the horizontal line and that the insertion of the static source χ does not carry momentum, unlike the one of γ_φ.

Using the relations $Z^c_{ij}\,\boldsymbol{\Gamma}_{jk} = i\delta_{ik}$ and $Z^c_{\bar I K}\,\boldsymbol{\Gamma}_{\bar K J} = i\delta_{IJ}\mathbf{1}$, where $\mathbf{1}$ is the identity matrix for the Dirac indices, we obtain the Nielsen identities for the **propagator functions**, which read

$$\begin{aligned} \partial_\xi Z^c_{ij} &= Z^c_{ik}\,\boldsymbol{\Gamma}_{\chi k\gamma_j} + \boldsymbol{\Gamma}_{\chi\gamma_i k}\,Z^c_{kj}, && (41)\\ \partial_\xi Z^c_{\bar I J} &= Z^c_{\bar I K}\boldsymbol{\Gamma}_{\chi\bar K\eta_J} + \boldsymbol{\Gamma}_{\chi\bar\eta_I K} Z^c_{\bar K J}, && (42) \end{aligned}$$

for bosons and fermions, respectively. Graphically, these identities can be represented by Fig. 4 after replacing the blobs with the χ insertion by their mirror images and exchanging the corresponding indices. For the **three-point functions** we have

$$\begin{aligned} -\partial_\xi\boldsymbol{\Gamma}_{\bar I J i} &= \boldsymbol{\Gamma}_{\chi\gamma_m\bar I J}\boldsymbol{\Gamma}_{mi} + \boldsymbol{\Gamma}_{\chi i\gamma_m}\boldsymbol{\Gamma}_{m\bar I J}\\ &+ \boldsymbol{\Gamma}_{\bar I K i}\boldsymbol{\Gamma}_{\chi\bar\eta_K J} + \boldsymbol{\Gamma}_{\bar I K}\boldsymbol{\Gamma}_{\chi i\bar\eta_K J} + \boldsymbol{\Gamma}_{\chi\bar I\eta_K}\boldsymbol{\Gamma}_{\bar K J i} + \boldsymbol{\Gamma}_{\chi i\bar I\eta_K}\boldsymbol{\Gamma}_{\bar K J}\,. \end{aligned} \quad (43)$$

We see that the gauge-dependent terms of the form of $\boldsymbol{\Gamma}_{\chi\gamma_i j}$ introduced by the propagators in Eq. (39) are exactly cancelled by the last term in the first line of Eq. (43), i.e. by the vertices alone. Therefore, the boxes are not necessary to remove the gauge-dependence of the internal self-energies. The identity for the **four-point functions** is

$$\begin{aligned} -\partial_\xi\boldsymbol{\Gamma}_{\bar I J\bar K N} &= \boldsymbol{\Gamma}_{\bar I J m}\,\boldsymbol{\Gamma}_{\chi\gamma_m\bar K N} + \boldsymbol{\Gamma}_{\bar I N m}\,\boldsymbol{\Gamma}_{\chi\gamma_m\bar K J} + \boldsymbol{\Gamma}_{\chi\gamma_m\bar I J}\,\boldsymbol{\Gamma}_{m\bar K N} + \boldsymbol{\Gamma}_{\chi\gamma_m\bar I N}\,\boldsymbol{\Gamma}_{m\bar K J}\\ &+ \boldsymbol{\Gamma}_{\bar I S}\,\boldsymbol{\Gamma}_{\chi\bar\eta_S J\bar K N} + \boldsymbol{\Gamma}_{\bar K S}\,\boldsymbol{\Gamma}_{\chi\bar I J\bar\eta_S N} + \boldsymbol{\Gamma}_{\chi\bar I J\bar K\eta_S}\,\boldsymbol{\Gamma}_{\bar S N} + \boldsymbol{\Gamma}_{\chi\bar I\eta_S\bar K N}\,\boldsymbol{\Gamma}_{\bar S J}\\ &+ \boldsymbol{\Gamma}_{\bar I J\bar K S}\,\boldsymbol{\Gamma}_{\chi\bar\eta_S N} + \boldsymbol{\Gamma}_{\bar I S\bar K N}\,\boldsymbol{\Gamma}_{\chi\bar\eta_S J} + \boldsymbol{\Gamma}_{\chi\bar I\eta_S}\,\boldsymbol{\Gamma}_{\bar S J\bar K N} + \boldsymbol{\Gamma}_{\chi\bar K\eta_S}\,\boldsymbol{\Gamma}_{\bar I J\bar S N}\,. \end{aligned} \quad (44)$$

We now distinguish between the different Green functions containing the source χ:

1. Terms of the form $\boldsymbol{\Gamma}_{\chi\gamma_i\bar I J}$ are present both in the gauge variation of the boxes (first line) and in the one of the vertices (first term). They cancel against each other in the sum (39) according to the pattern

$$\underbrace{\boldsymbol{\Gamma}_{\chi\gamma_m\bar I J}\;\boldsymbol{\Gamma}_{mi}\;Z^c_{ij}\;\boldsymbol{\Gamma}_{j\bar K N}}_{-(\partial_\xi\boldsymbol{\Gamma}_{\bar I j i})\;Z^c_{ij}\;\boldsymbol{\Gamma}_{j\bar K N}} - \underbrace{i\,\boldsymbol{\Gamma}_{\chi\gamma_m\bar I J}\;\boldsymbol{\Gamma}_{m\bar K N}}_{i\,\partial_\xi\boldsymbol{\Gamma}_{\bar I J\bar K N}} = 0\ ,$$

where we have specified which part of Eq. (39) generates each term.

2. The factors containing $\mathbf{\Gamma}_{\chi i\bar{\eta}_K J}$ in the second line of Eq. (43) and $\mathbf{\Gamma}_{\chi\bar{\eta}_S J\bar{K}N}$ (the whole second line of Eq. (44)) always multiply a two-point function of the external fermions like $\mathbf{\Gamma}_{\bar{I}J}$. When they are contracted with the external spinors, these terms vanish, as a consequence of Eq. (37).

3. The remaining terms contain Green functions of the kind $\mathbf{\Gamma}_{\chi\bar{\eta}_I J}$ and $\mathbf{\Gamma}_{\chi\bar{I}\eta_J}$ which multiply vertices and boxes in Eqs. (43) and (44), respectively. As we will see in a moment, they are cancelled by the LSZ factors.

Adding together the various pieces, the gauge-parameter variation of the on-shell truncated Green function can be expressed in terms of the truncated function itself:

$$-\partial_\xi Z^{trunc}_{\bar{I}J\bar{K}N}|_{on-shell} = \mathbf{\Gamma}_{\chi\bar{I}\eta_S}\, Z^{trunc}_{\bar{S}J\bar{K}N} + \mathbf{\Gamma}_{\chi\bar{K}\eta_S}\, Z^{trunc}_{\bar{I}J\bar{S}N} + Z^{trunc}_{\bar{I}S\bar{K}N}\,\mathbf{\Gamma}_{\chi\bar{\eta}_S J} + Z^{trunc}_{\bar{I}J\bar{K}S}\,\mathbf{\Gamma}_{\chi\bar{\eta}_S N}, \quad (45)$$

according to the usual form for the Nielsen identities. Of course, this on-shell factorization holds in general for any amputated Green function, as it follows from the gauge independence of the S-matrix.

We are now ready to apply the LSZ reduction formula. The gauge variation of the factor $\widetilde{Z}$ can be computed from Eq. (36) and Eq. (37) using the Nielsen identities for the two-point functions and the gauge-independence of the asymptotic spinors $u_{as,I}$. We then obtain

$$\lim_{on-shell} \partial_\xi \widetilde{Z}^{1/2}_{\bar{I}J}\, u_{as,J} = \mathbf{\Gamma}_{\chi\bar{I}\eta_S} \widetilde{Z}^{1/2}_{\bar{S}J}\, u_{as,J}, \quad (46)$$

where $\mathbf{\Gamma}_{\chi\bar{I}\eta_S}$ is calculated on-shell, from which the final cancellation of the gauge-dependence follows.

If some of the β_i^ξ do not vanish, the cancellations do not operate any longer and the amplitude turns out to be gauge parameter dependent [16]. An explicit example has been considered in [39], for the W decay into quarks: if the CKM counterterm is gauge-dependent, the amplitude depends on the gauge parameters too. On the other hand, the above proof relies neither on a specific choice of renormalization of the unphysical parameters, nor on the regularization scheme adopted.

8 Summary

We have introduced the Nielsen identities of the SM and used the problem of the definition of mass as a demonstrative example. In this context we have obtained some new results: we have proven to all orders in perturbation theory the gauge-parameter independence of the complex pole associated to any physical particle of the SM. We have considered the cases of the vector bosons, scalars and fermions in great generality, allowing for arbitrary mixing patterns. Particular attention has been paid to the case of the W boson, which is simpler because of the absence of mixing and has been chosen to illustrate some features common to all cases. Most of the proofs hold without modifications also in some extensions of the SM, like non-supersymmetric two-Higgs-doublet models.

We have derived identities for the gauge-dependence of the tadpoles and of all the two-point functions of the SM, both for bosons and fermions, as well as for vertices and boxes involving external fermions. Using these expressions, we have shown the explicit mechanism of gauge cancellations which leads to gauge-independent four fermion amplitudes, independently of the perturbative expansion, in the most general case of fermion and boson mixings and of CP violation. The formalism introduced in this paper, supplemented by the material given in App. A (the Lagrangian involving the BRST sources), should allow for a very simple derivation of the Nielsen identities for any proper Green function in the electroweak SM and in QCD.

We have also extensively discussed the renormalization of the Nielsen identities, considering as a starting point the most general case in which the regularization is not invariant and the renormalization breaks the underlying symmetry which generates the identities themselves. In that case the identities are deformed by new terms, which we have identified in full generality and computed in a few cases of particular interest. We have also derived new results concerning the infrared-finiteness of the W pole mass and the photon two-point function at $q^2 = 0$ in the SM. For completeness, we report in App. B the expressions for the fermionic one-loop self-energies in a generic R_ξ gauge.

In conclusion, the formalism of the Nielsen identities can be useful in various applications, at the conceptual level (e.g. for the identification of gauge-independent quantities such as invariant charges [7] and the definition of gauge-independent renormalized parameters [39]) as well as at the practical level (in particular, for checks of higher order calculations): it deserves to be better known to theorists.

Acknowledgments

We are grateful to A. Sirlin and W. Zimmermann for interesting discussions and to M. Steinhauser for useful communications and a careful reading of the manuscript. This work has been supported in part by the Bundesministerium für Bildung und Forschung under contract 06 TM 874 and by the DFG project Li 519/2-2.

A. Nielsen identities for pedestrians

The aim of this Appendix is to review very briefly the formalism of Slavnov-Taylor Identities (STI) in the case of the Nielsen identities and to provide some material necessary for the explicit calculation of the Green functions involving the BRST sources. For a non-expert introduction to the STI for specific physical amplitudes, we refer to [41]. First, we recall that in our conventions the gauge-fixing term in the SM Lagrangian is given by

$$\mathcal{L}_{GF} \;=\; -\frac{1}{2\xi_A}\,(\partial^\mu A_\mu)^2 - \frac{1}{2\xi_Z^{(1)}}\left(\partial^\mu Z_\mu - \frac{\xi_Z^{(2)}\sqrt{g'^2+g^2}}{2}\,v\,G^0\right)^2$$

$$- \quad \frac{1}{\xi_W^{(1)}} \left| \partial^\mu W_\mu^+ - \frac{i\xi_W^{(2)}\, g}{2}\, v\, G^+ \right|^2 - \frac{1}{2\xi_g} \left(\partial^\mu G_\mu^b \right)^2 . \tag{A1}$$

We always set $\xi_{W,Z} \equiv \xi_{W,Z}^{(1)} = \xi_{W,Z}^{(2)}$, i.e. we confine ourselves to the restricted 't Hooft gauge. Our starting point is the complete generating vertex functional $\mathbf{\Gamma}^c$, which generates the one-particle-irreducible Green functions. In order to simplify the structure of the STI, it is convenient to introduce for linear gauge-fixings a *reduced* generating functional $\mathbf{\Gamma}$ (sometimes indicated by $\hat{\mathbf{\Gamma}}$ in the literature), which differs from $\mathbf{\Gamma}^c$ by a local term, corresponding to the gauge-fixing part of the Lagrangian:

$$\mathbf{\Gamma} = \mathbf{\Gamma}^c - \int d^4x \; \mathcal{L}_{GF}. \tag{A2}$$

In practice, the STI obtained from $\mathbf{\Gamma}$ coincide with the STI obtained from $\mathbf{\Gamma}^c$ after implementation of the ghost equation [33]. Of course, one should keep in mind that the Green functions involving unphysical fields generated by $\mathbf{\Gamma}$ coincide with the ones generated by $\mathbf{\Gamma}^c$ only up to constant terms. For example, one has $\mathbf{\Gamma}^{(0)}_{W_\mu W_\nu} = \mathbf{\Gamma}^{c(0)}_{W_\mu W_\nu} + p^\mu p^\nu/\xi_W$ and $\mathbf{\Gamma}^{(0)}_{G^+G^-} = \mathbf{\Gamma}^{c(0)}_{G^+G^-} + \xi_W M_W^2$ at the tree level, while the difference at higher orders depends only on the renormalization of the W field and of the gauge parameters. As we have eliminated the classical gauge-fixing, it is clear that $\mathbf{\Gamma}_{W^+G^-} \neq 0$ already at tree level.

The invariance of the action under BRST transformations implies the STI for the functional $\mathbf{\Gamma}$ (see for ex. [33]),

$$\mathcal{S}_\Gamma = \int d^4x \sum_\varphi \left[\frac{\delta \mathbf{\Gamma}}{\delta \gamma_\varphi} \frac{\delta}{\delta \varphi} + \frac{\delta \mathbf{\Gamma}}{\delta \varphi} \frac{\delta}{\delta \gamma_\varphi} \right]; \qquad \mathcal{S}_\Gamma \, \mathbf{\Gamma} = 0, \tag{A3}$$

where φ stands for any of the quantum fields in the SM Lagrangian (gauge fields, scalars, ghosts, and fermions) and γ_φ is the BRST source associated to φ, which is coupled to the BRST variation of φ in the classical action. In the case of a fermion f_I the spinorial source is denoted by η_I. $\mathcal{S}_\Gamma$ is the Slavnov-Taylor operator. By functional differentiation of Eq. (A3) wrt some SM fields one gets the Slavnov-Taylor Identities (STI). Electric and ghost charge conservation, as well as Lorentz invariance, should be taken into account, according to the examples given in the text.

In order to obtain the Nielsen identities for the gauge parameter dependence of irreducible Green functions, we have to consider the case of extended BRST symmetry [1], which involves also the transformation of the gauge parameters; Eq. (A3) takes then the form

$$\mathcal{S}_\Gamma \, \mathbf{\Gamma} + 2 \sum_i \chi_i \, \partial_{\xi_i} \mathbf{\Gamma} = 0, \tag{A4}$$

from which Eq. (1) follows after setting $\chi = 0$. In the fermionic sector the expressions are slightly complicated by the anticommutation relations and the Nielsen identity becomes

$$\partial_\xi \mathbf{\Gamma}^{fer} = \sum_I \left[\frac{\mathbf{\Gamma} \overleftarrow{\delta}}{\delta \psi_I} \frac{\overrightarrow{\delta}\, \partial_\chi \mathbf{\Gamma}}{\delta \bar{\eta}_I} - \frac{\partial_\chi \mathbf{\Gamma} \overleftarrow{\delta}}{\delta \psi_I} \frac{\overrightarrow{\delta}\, \mathbf{\Gamma}}{\delta \bar{\eta}_I} + (\psi_I \leftrightarrow \eta_I) \right], \tag{A5}$$

where $\partial_\chi = \partial/\partial\chi$ and the arrows indicate the direction in which the functional derivative wrt the fermionic field acts (this is important for anticommuting fields).

We have seen that both the Nielsen identities and the STI contain Green functions involving the BRST sources γ_φ and η_f (for fermions) associated to the various fields of the SM. If we want to compute these Green functions at a given order in perturbation theory, we need to know how the sources are coupled to the fields. To this end, we now give the complete action involving the BRST sources, which can be useful as a reference and to obtain the Feynman rules necessary for actual calculations involving γ_φ and η_f. Apart from the well-known Feynman rules of the SM (see for instance the second paper in [29]), nothing else is needed to evaluate the unconventional objects that appear in the identities. Using the convention $Z_\mu = c_W W^3_\mu + s_W B_\mu$, where W^3_μ, B_μ are the third component of the triplet of $SU(2)_L$ and the $U_Y(1)$ gauge boson, respectively, we have

$$\begin{aligned}
\mathcal{L}_{BRST} &= \gamma^\mu_A \left\{\partial_\mu c^A + ie\left[W^+_\mu c^- - W^-_\mu c^+\right]\right\} + \gamma^\mu_Z \left\{\partial_\mu c^Z - ie\frac{c_W}{s_W}\left[W^+_\mu c^- - W^-_\mu c^+\right]\right\} \\
&+ \gamma^{\mp\mu}_W \left\{\partial_\mu c^\pm \mp ieW^\pm_\mu \left(c^A - \frac{c_W}{s_W}c^Z\right) \pm iec^\pm\left[A_\mu - \frac{c_W}{s_W}Z_\mu\right]\right\} \\
&+ \gamma^{a\mu}\left\{\partial_\mu c^a - g_s f^{abc} G^b_\mu c^c\right\} + \gamma_{c^Z}\left\{-ie\frac{c_W}{s_W}c^+c^-\right\} + \gamma_{c^A}\left\{iec^+c^-\right\} \\
&+ \gamma_{c^\mp}\left\{\mp\frac{ie}{2}c^\pm\left(c^A - \frac{c_W}{s_W}c^Z\right)\right\} + \gamma_{c^a}\left\{\frac{g_s}{2}f^{abc}c^bc^c\right\} \\
&+ \gamma^H\left\{\frac{ig}{2}\left[G^+c^- - G^-c^+\right] + \frac{g}{2c_W}G^0c^Z\right\} \\
&+ \gamma^\mp\left\{\pm\frac{ig}{2}\left[H + v \pm iG^0\right]c^\pm \mp ieG^\pm\left(c^A - \frac{c_W^2 - s_W^2}{2c_Ws_W}c^Z\right)\right\} \\
&+ \gamma^0\left\{\frac{g}{2}\left[G^+c^- + G^-c^+\right] - \frac{g}{2c_W}\left(H+v\right)c^Z\right\} \\
&+ i\left(\overline{\eta}_\nu, \overline{\eta}^L_l\right)\begin{pmatrix} \frac{g}{\sqrt{2}}\, l^L\, c^+ + \frac{g}{2}\frac{c^Z}{c_W}\nu \\ \frac{g}{\sqrt{2}}\nu c^- - e\left[Q_lc^A + \left(\frac{1}{2s_W} + Q_ls_W\right)\frac{c^Z}{c_W}\right]l^L \end{pmatrix} \\
&+ i\left(\overline{\eta}^L_u, \overline{\eta}^L_d\right)\begin{pmatrix} \frac{gV_{ud}}{\sqrt{2}}d^Lc^+ - e\left[Q_uc^A - \left(\frac{1}{2s_W} - Q_us_W\right)\frac{c^Z}{c_W}\right]u^L + g_s\frac{\lambda^a}{2}u^Lc_a \\ \frac{gV^*_{ud}}{\sqrt{2}}u^Lc^- - e\left[Q_dc^A + \left(\frac{1}{2s_W} + Q_ds_W\right)\frac{c^Z}{c_W}\right]d^L + g_s\frac{\lambda^a}{2}d^Lc_a \end{pmatrix} \\
&- i\,\overline{\eta}^R_l\left\{eQ_l\left(c^A + \frac{s_W}{c_W}c^Z\right)l^R\right\} + i\,\overline{\eta}^R_u\left\{-e\,Q_u\left(c^A + \frac{s_W}{c_W}c^Z\right)u^R + g_s\frac{\lambda^a}{2}u^Rc_a\right\} \\
&+ i\,\overline{\eta}^R_d\left\{-e\,Q_d\left(c^A + \frac{s_W}{c_W}c^Z\right)d^R + g_s\frac{\lambda^a}{2}d^Rc_a\right\} + \text{h.c.}
\end{aligned} \tag{A6}$$

where λ^a are the Gell-Mann matrices, R and L indicate the right and left-handed components of the fermion fields, and $s_W = \sin\theta_W$, $c_W = \cos\theta_W$. The hermitian conjugate for the fermionic part is added at the end. The ghost charge of the various sources, which is important in writing the STI, can be inferred by Eq. (A6), assigning a number +1 to the ghosts and requiring $\mathcal{L}$ to be ghost charge neutral. We have introduced two different sources γ_A, γ_Z for the BRST transformations of the A_μ and Z_μ, respectively. This is not strictly necessary since the composite operators coupled to γ_A, γ_Z coincide up to (trivial) terms linear in the ghost fields

$$\begin{aligned}\frac{\delta\Gamma^{(0)}}{\delta\gamma^A_\mu} &= -s_W c_W \partial_\mu c^Z + c_W^2 \partial_\mu c^A + s_W \frac{\delta\Gamma^{(0)}}{\delta\gamma^3_\mu} \\ \frac{\delta\Gamma^{(0)}}{\delta\gamma^Z_\mu} &= s_W^2 \partial_\mu c^Z - s_W c_W \partial_\mu c^A + c_W \frac{\delta\Gamma^{(0)}}{\delta\gamma^3_\mu}\end{aligned} \tag{A7}$$

where γ^3_μ is the source of the BRST transformation of the third component of the gauge boson triplet. In abelian gauge theories, for example, it is not necessary to associate a BRST source to the gauge field. On the other hand, our choice is convenient in order to have compact and simple expressions for the Nielsen identities.

The last ingredient for the calculation of the Green functions involving the source χ, characteristic of the Nielsen identities, are the couplings of χ with the other fields. There is a source χ_i associated to any gauge parameter ξ_i [2]. The relevant Lagrangian takes the form:

$$\begin{aligned}\mathcal{L}_\chi &= -\frac{\chi_g}{2\xi_g}\,\bar{c}^a\,\partial_\mu G^{a,\mu} - \frac{\chi_A}{2\xi_A}\,\bar{c}^A\,\partial_\mu A^\mu - \frac{\chi_Z}{2\xi_Z}\,\bar{c}^Z\,(\partial_\mu Z^\mu + \xi_Z M_Z G_0) \\ &- \frac{\chi_W}{2\xi_W}\left[\bar{c}^+\left(\partial^\mu W^-_\mu - i\xi_W M_W G^-\right) + \bar{c}^-\left(\partial^\mu W^+_\mu + i\xi_W M_W G^+\right)\right]\end{aligned} \tag{A8}$$

B. Gauge dependence of the fermionic self-energies

In this appendix we present the explicit gauge-parameter dependence of the one-loop fermionic unrenormalized self-energies in the SM. We consider the most general case of mixing and define the fermionic self-energy Σ_{ij} as $+i$ times the standard Feynman amplitude for the transition $j \to i$ and extract a factor g^2. The expressions in the 't Hooft-Feynman gauge ($\xi_i = 1$) can be found, for example, in Ref. [42]. At the one-loop level, instead of Eq. (30), we can use the decomposition

$$\Sigma_{ij}(p) = \Sigma^L_{ij}(p^2)\;\not{p}\,P_L + \Sigma^R_{ij}(p^2)\;\not{p}\,P_R + \Sigma^S_{ij}(p^2)\,(m_i P_L + m_j P_R)\,.$$

The individual components of the self-energies are then given in an arbitrary gauge by

$$\Sigma^S_{ij} \;=\; \Sigma^S_{ij}|_{\xi=1} + (\xi_\gamma - 1)\,\delta_{ij}\,s_W^2\,Q_i^2\,b_{\gamma i} + (\xi_W - 1)\sum_k \lambda^{ij}_k\,\frac{m_k^2}{2}\,c_{Wk}$$

[2]Having set the two gauge parameters $\xi_i^{(1,2)}$ equal to each other, we can work with only one source χ_i. This differs slightly from the procedure adopted in [39], where two distinct sources $\chi_i^{(1,2)}$ were kept.

$$+ \quad (\xi_Z - 1)\,\frac{\delta_{ij}}{c_W^2}\left[\ell_i\, r_i\, b_{Zi} + \left(\ell_i\, r_i\, \xi_Z\, M_Z^2 + \frac{m_i^2}{4}\right) c_{Zi}\right] \tag{B1}$$

$$\begin{aligned}
\Sigma_{ij}^L &= \Sigma_{ij}^L|_{\xi=1} \\
&+ (\xi_\gamma - 1)\,\delta_{ij}\,\frac{s_W^2}{2}\,Q_i^2\left[p^2(1-x_i)^2 c_{\gamma i} - (1-x_i)\alpha_\gamma - (1+x_i) b_{\gamma i}\right] \\
&+ (\xi_W - 1)\sum_k \frac{\lambda_k^{ij}}{4}\left[p^2(1-3x_k)c_{Wk} - b_{Wk} - \xi_W M_W^2 c_{Wk} - \alpha_W\right] \\
&+ (\xi_Z - 1)\,\frac{\delta_{ij}}{2c_W^2}\left\{p^2\, c_{Zi}\left[\ell_i^2(1-x_i)^2 - \frac{x_i}{4}(1+x_i)\right] - \left(\ell_i^2(1-x_i) + \frac{x_i}{4}\right)\alpha_Z\right. \\
&- \left.\left[\ell_i^2(1+x_i) - \frac{x_i}{4}\right]\left(b_{Zi} + \xi_Z\, M_Z^2\, c_{Zi}\right)\right\}
\end{aligned} \tag{B2}$$

$$\begin{aligned}
\Sigma_{ij}^R &= \Sigma_{ij}^R|_{\xi=1} \\
&+ (\xi_\gamma - 1)\,\delta_{ij}\,\frac{s_W^2}{2}\,Q_i^2\left[p^2(1-x_i)^2 c_{\gamma i} - (1-x_i)\alpha_\gamma - (1+x_i) b_{\gamma i}\right] \\
&- (\xi_W - 1)\sum_k \lambda_k^{ij}\,\frac{m_i m_j}{4p^2}\left[\alpha_W - b_{Wk} + (m_k^2 + p^2 - \xi_W M_W^2)c_{Wk}\right] \\
&+ (\xi_Z - 1)\,\frac{\delta_{ij}}{2c_W^2}\left\{p^2\, c_{Zi}\left[r_i^2(1-x_i)^2 - \frac{x_i}{4}(1+x_i)\right] - \left(r_i^2(1-x_i) + \frac{x_i}{4}\right)\alpha_Z\right. \\
&- \left.\left[r_i^2(1+x_i) - \frac{x_i}{4}\right]\left(b_{Zi} + \xi_Z\, M_Z^2\, c_{Zi}\right)\right\}
\end{aligned} \tag{B3}$$

where we have used the following notation for the n-dimensional integrals $(i, j = \gamma, Z^0, W, f)$

$$\begin{aligned}
\alpha_i &= i\mu^{4-n}\int \frac{d^n k}{(2\pi)^n}\frac{1}{[k^2 - m_i^2][k^2 - \xi_i\, m_i^2]} \\
b_{ij} &= i\mu^{4-n}\int \frac{d^n k}{(2\pi)^n}\frac{1}{[k^2 - m_i^2][(k+p)^2 - m_j^2]} \\
c_{ij} &= i\mu^{4-n}\int \frac{d^n k}{(2\pi)^n}\frac{1}{[k^2 - m_i^2][k^2 - \xi_i m_i^2][(k+p)^2 - m_j^2]}.
\end{aligned} \tag{B4}$$

We have also used $x_i = m_i^2/p^2$, while $\ell_i = I_i^3 - Q_i s_W^2$ and $r_i = -Q_i s_W^2$ are the left and right-handed couplings of the fermion flavor i and Q_i and $I_i^3 = \pm\frac{1}{2}$ its electric charge and isospin. In the case of quarks, the mixing matrix factor λ_k^{ij} equals $V_{ik}V_{jk}^*$, where V is the CKM matrix, if i, j (k) are up (down) quarks and $\lambda_k^{ij} = V_{ki}^* V_{kj}$ if the opposite is true. For leptons with massless neutrinos $\lambda_k^{ij} = \delta_{ij}\,\delta_{k\nu_i}$ or $\delta_{ij}\,\delta_{kl_i}$, i.e. there is no mixing. The gluon exchange diagrams can be obtained from the photonic ones setting $Q_i = 1$ and multiplying by the color factor C_F. Notice that α_γ and $c_{\gamma i}$ are infrared divergent and an infrared regulator (like a photon mass) should be introduced. Of course, the infrared divergences cancel out in Eqs.(B2-B3). It is straightforward to verify that in the diagonal case the mass counterterm, $\delta m_i/m_i = \Sigma_{ii}^S(m_i^2) + \frac{1}{2}\Sigma_{ii}^L(m_i^2) + \frac{1}{2}\Sigma_{ii}^R(m_i^2) + T_i$, where T_i is the tadpole contribution, is independent of the gauge parameters. From the off-diagonal parts of Eqs.(B1-B3) it is easy to derive some of the results of Ref. [39] on the gauge dependence of the CKM counterterm.

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